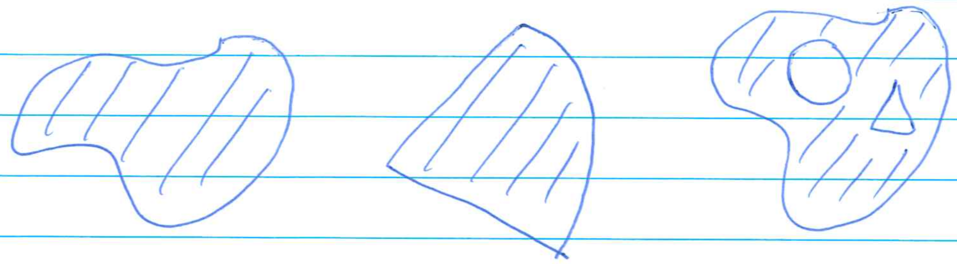
2020 A
Fall 2020/21

Lecture 3

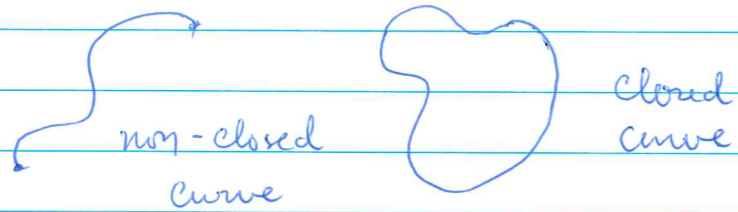
- Double $\iint$ over regions.

Some examples of regions:

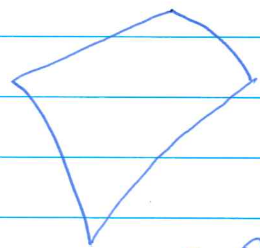


A region consists of interior points and boundary points. Only bounded regions are considered. They can be contained in some rectangles.

a smooth curve:



a piecewise smooth curve: there are 4 vertices which admits no tangents.



A region (or domain) is a set in $\mathbb{R}^2$ bounded by one or several piecewise smooth closed, simple curves.

Let f be a function defined on a set $S \subset \mathbb{R}^2$. Define its extension to $\mathbb{R}^2$ by

$$\tilde{f}(x) = \begin{cases} f(x), & x \in S \\ 0, & x \notin S. \end{cases}$$

When S is bounded, define

$$\iint_S f \stackrel{\text{def}}{=} \iint_{R_0} \tilde{f}$$

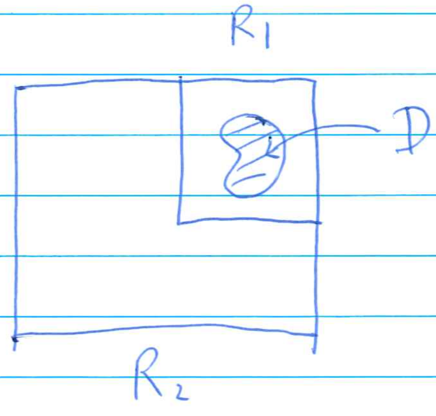
whenever R_0 is a rectangle containing S (S is bdd, so always possible.) the integral on the right hand sides make sense when $\tilde{f}$ is integrable.

In particular, when S is a region and f is piecewise continuous in S, we know that $\tilde{f}$ is piecewise continuous in any R_0, hence the RHS makes sense.

According to the following theorem, this $\int$ on the RHS also is independent of the chosen R_0.

<u>Theorem 6</u> Let R_1, R_2 be two rectangles containing D. The

$$\iint_{R_1} \tilde{f} = \iint_{R_2} \tilde{f}$$



Prof. <u>Step 1</u> Assume $R_1 \subset R_2$.

Given $\varepsilon > 0$, $\exists\ \delta_2$ s.t.

$$\left| \iint_{R_2} \tilde{f} - S(\tilde{f}, \mathring{P}) \right| < \varepsilon ,$$

for P on R_2, $\|P\| < \delta_2$. Also, $\exists\ \delta_1$

$$\left| \iint_{R_1} \tilde{f} - S(\tilde{f}, \mathring{Q}) \right| < \varepsilon ,$$

$\forall\, Q$ on R_1, $\|Q\| < \delta_1$.

We choose P such that the boundary of R_1 lies on the boundaries of the subrectangles of P and Q the subrectangles of P inside R_1. Taking the tags to be the center of the subrectangles,

$$S(\tilde{f}, \dot{P}) = S(\tilde{f}, \dot{Q}) \quad \text{since } \tilde{f} = 0 \text{ on } R_2 \setminus R_1.$$

$$\therefore \left| \iint_{R_2} \tilde{f} - \iint_{R_1} \tilde{f} \right| = \left| \iint_{R_2} \tilde{f} - S(\tilde{f}, \dot{P}) + S(\tilde{f}, \dot{Q}) - \iint_{R_1} \tilde{f} \right|$$

$$\leq \left| \iint_{R_2} \tilde{f} - S(\tilde{f}, \dot{P}) \right| + \left| S(\tilde{f}, \dot{Q}) - \iint_{R_1} \tilde{f} \right|$$

$$< 2\varepsilon$$

$\forall P, \|P\| < \delta_1, \delta_2$. So, $\displaystyle\iint_{R_2} \tilde{f} = \iint_{R_1} \tilde{f}$.

<u>step 2</u> when $D \subset R_1$, $D \subset R_2$, we have

$$D \subset R_1 \cap R_2 \subset R_1, R_2$$

so $$\iint_{R_1} \tilde{f} = \iint_{R_1 \cap R_2} \tilde{f} = \iint_{R_2} \tilde{f} \quad \text{by step 1.} \quad \square$$

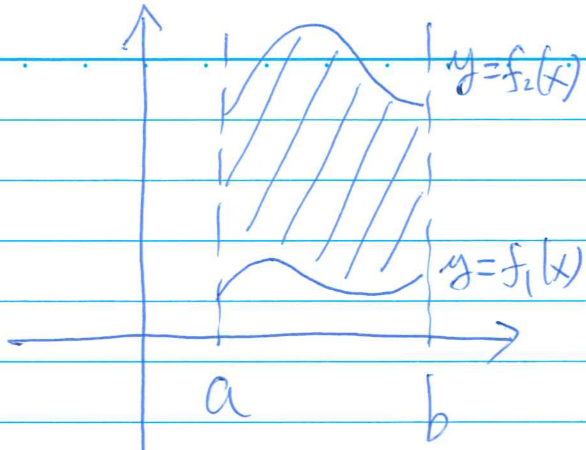
How to compute $\displaystyle\iint_D f$?

Important special case :

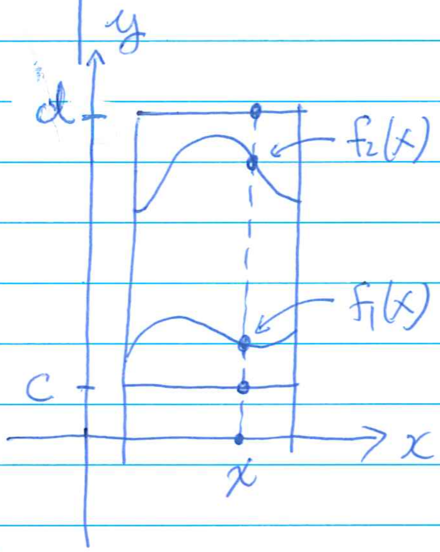
D is bdd by $x = a$, $x = b$, $y = f_1(x)$, $y = f_2(x)$ where $f_1(x) \leq f_2(x)$, $x \in [a, b]$.

<u>Theorem 7</u> For D described above,

$$\iint_D f = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx$$



Pf: Let $D \subset [a,b]\times[c,d]$



$$\iint_D f = \iint_{[a,b]\times[c,d]} \tilde{f}$$

$$= \int_a^b \int_c^d \tilde{f}(x,y)\,dy\,dx$$

(∵ Fubini's)

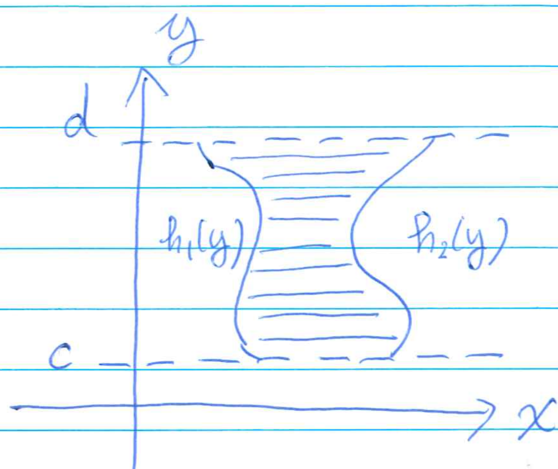
$$= \int_a^b \left(\int_c^{f_1(x)} \tilde{f}(x,y)\,dy \right) dx + \int_a^b \left(\int_{f_1(x)}^{f_2(x)} \tilde{f}(x,y)\,dy \right) dx + \int_a^b \left(\int_{f_2(x)}^{d} \tilde{f}(x,y)\,dy \right) \cdot$$

$$= 0 + \int_a^b \int_{f_1(x)}^{f_1(x)} \tilde{f}(x,y)\,dy\,dx + 0 \qquad (\because \tilde{f} = 0 \text{ there})$$

$$= \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx \qquad (\because \tilde{f} = f \text{ there}).$$

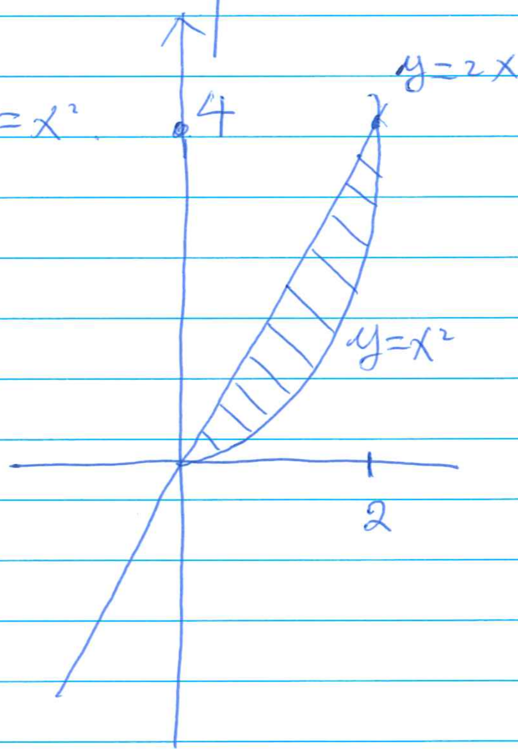
Reversing order, we have

$$\iint_D f = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$



e.g. 5 $\iint_D (2y+1)$ when D is the region bdd by $y=2x$, and $y=x^2$.

Here $f_1(x)=x^2$, $f_2(x)=2x$



$$\therefore \iint_D (2y+1) = \int_0^2 \int_{x^2}^{2x} (2y+1)\,dy\,dx$$

$$= \int_0^2 (y^2+y)\Big|_{x^2}^{2x} dx$$

$$= \int_0^2 (3x^2+2x-x^4)\,dx$$

$$= \frac{28}{5}.$$

Another way: $h_1(y)=\frac{y}{2}$, $h_2(y)=\sqrt{y}$

$$\iint_D (2y+1) = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (2y+1)\,dx\,dy$$

$$= \int_0^4 (2y+1) \int_{\frac{y}{2}}^{\sqrt{y}} 1 . dx\, dy$$

$$= \int_0^4 (2y+1)\left(\sqrt{y} - \frac{y}{2}\right) dy$$

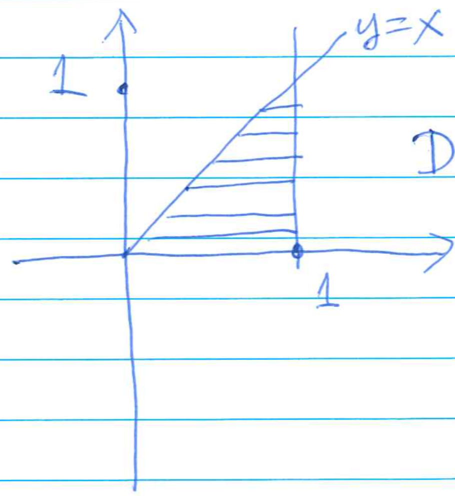
$$\vdots$$

$$= \frac{28}{5} \ \#$$

e.g 6. Find $\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$.

Don't know how to work on $\int \frac{\sin x}{x}$, so need to reverse the order of iteration. D is given by $0 \le y \le 1$, $h_1(y) = y$, $h_2(y) = 1$

$$\therefore \int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy = \iint_D \frac{\sin x}{x}$$



$$= \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

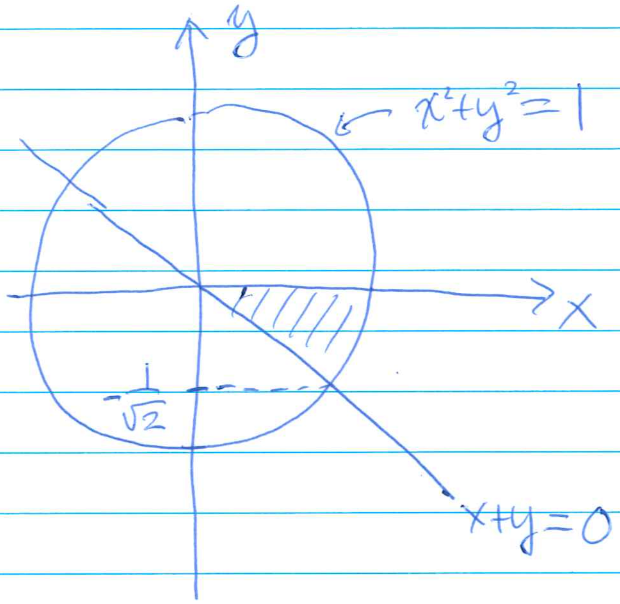
$$= \int_0^1 \frac{\sin x}{x} \int_0^x 1\, dy\, dx$$

$$= \int_0^1 \frac{\sin x}{x}\, x\, dx$$

$$= \int_0^1 \sin x\, dx$$

$$= 1 - \cos 1 . \#$$

e.g. 7. $\iint_D x$ where D is the domain bdd by $y=0$, $x+y=0$, the unit circle, $x \geq 0$.



D can be described as $-\frac{1}{\sqrt{2}} \leq y \leq 0$, $h_1(y) = -y$, $h_2(y) = \sqrt{1-y^2}$.

$$\therefore \iint_D x = \int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-y}^{\sqrt{1-y^2}} x \, dx \, dy$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{0} \frac{1}{2}(1-y^2-y^2)\,dy$$

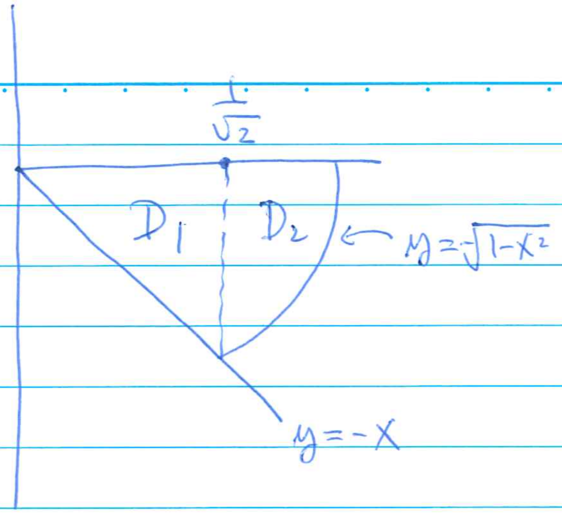
$$= \frac{1}{2}\left(y - \frac{2}{3}y^3\right)\Big|_{-\frac{1}{\sqrt{2}}}^{0}$$

$$= \frac{1}{3\sqrt{2}}.$$

One may use the other way, but then we need to break up the domain:

$D_1 : 0 \le x \le \frac{1}{\sqrt{2}}$,

$f_1(x) = -x$, $f_2(x) = 0$.



$D_2 : \frac{1}{\sqrt{2}} \le x \le 1$,

$f_1(x) = -\sqrt{1-x^2}$, $f_2(x) = 0$.

Then

$$\iint_D x = \iint_{D_1} x + \iint_{D_2} x$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \int_{-x}^{0} x \, dy \, dx + \int_{\frac{1}{\sqrt{2}}}^{1} \int_{-\sqrt{1-x^2}}^{0} x \, dy \, dx$$

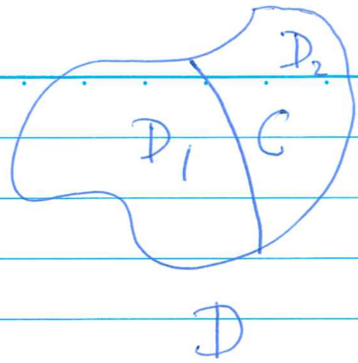
$$= \int_0^{\frac{1}{\sqrt{2}}} x^2 \, dx + \int_{\frac{1}{\sqrt{2}}}^{1} x\sqrt{1-x^2} \, dx$$

$$= \frac{1}{3\sqrt{2}} \quad \#$$

In the above example we have used the "decomposition principle" for integral. We formulate it as follows.

<u>Theorem 7</u> Let C be a piecewise smooth curve dividing D into two regions D_1 and D_2. Then

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f.$$



Introduce the useful notion of a characteristic function.

Let S be a nonempty set $\subseteq \mathbb{R}^2$ (or $\mathbb{R}^n$). its characteristic function:

$$\chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S. \end{cases}$$

Note

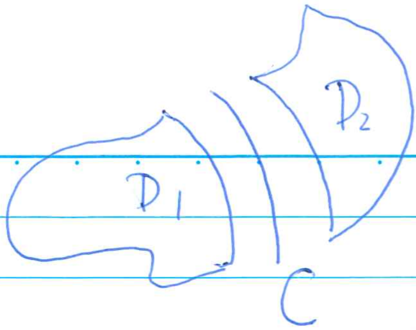
- $\chi_A \le \chi_B$ iff $A \subset B$
- $\chi_{A \cup B} \le \chi_A + \chi_B$ and "$=$" iff $A \cap B = \phi$
- $\chi_A \chi_B = \chi_{A \cap B}$

characteristic fcns turn set operations into function operations.

Note that $$\iint_D f = \iint_{R_0} \tilde{f} \chi_D.$$

"Pf of Theorem 7". We have

$$\chi_{D_1} + \chi_{D_2} = \chi_D + \chi_C$$



C has been counted twice.

$$\therefore \quad \tilde{f}\chi_{D_1} + \tilde{f}\chi_{D_2} = \tilde{f}\chi_D + \tilde{f}\chi_C$$

By linearity,

$$\iint_{R_0} \tilde{f}\chi_{D_1} + \iint_{R_0} \tilde{f}\chi_{D_2} = \iint_{R_0} \tilde{f}\chi_D + \iint_{R_0} \tilde{f}\chi_C$$

ie

$$\iint_{D_1} f + \iint_{D_2} f = \iint_{D} f + \iint_{C} f.$$

Now, thm 8 asserts $\iint_C f = 0$, we get it.

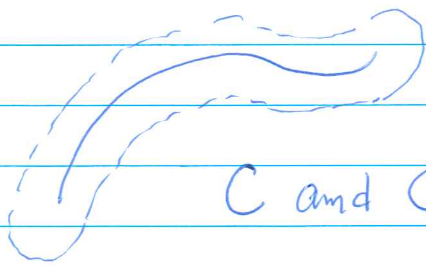
<u>Theorem 8</u> Let C be a piecewise smooth curve.

$$\iint_C f = 0, \text{ for integrable } f.$$

"Pf" Let $C_\delta = \{ p \in \mathbb{R}^2, \text{ distance of } p \text{ to } C \le \delta \}$

$C \subset C_\delta$

$\Rightarrow \chi_C \le \chi_{C_\delta}$



C and C_δ

assume

$-M \le f \le M$

$$-M\chi_{C_\delta} \le -M\chi_C \le f\chi_C \le M\chi_C \le M\chi_{C_\delta}$$

$$\therefore \quad -M\iint \chi_{C_\delta} \le \iint f\chi_C \le M\iint \chi_{C_\delta}.$$

It suffic to show

$$\lim_{\delta\to 0}\iint \chi_{C_\delta} = 0.$$

C_δ in ideal case

But $$\iint \chi_{C_\delta} = \iint_{C_\delta} 1\, dA$$

$$= \text{area of } C_\delta$$

$$\le \text{length of } C \times 2\delta + \pi\delta^2$$

$$\therefore \lim_{\delta\to 0}\iint \chi_{C_\delta} = 0, \text{ done. } \#$$